J. Cons. int. Explor. Mer, 40(1): 47–52. 1982.

The effect of unevenly distributed catches on stock-size estimates using Virtual Population Analysis (Cohort Analysis)

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The relative error in the value of the stock size calculated by Virtual Population Analysis (Cohort Analysis) that results from assuming that catches are evenly distributed throughout the year when, in fact, they are not is examined using both analytic methods and simulation. A measure of the intra-year distribution of catches called the balance is defined, and a relationship between the relative error in the calculated values of stock size, the balance, the natural mortality rate, and the rate of exploitation is derived.

Introduction

Virtual Population Analysis (VPA) is widely used in the assessment of exploited marine fisheries. This technique as currently used does not involve virtual populations, although an equivalent method introduced by Gulland (1965) does indeed do so. Perhaps a more appropriate term for this technique would be Backward Cohort Analysis, but the traditional phrase will be employed in the subsequent discussion. Virtual Population Analysis is a stepwise procedure to solve, backward in time, for the stock size and the fishing mortality rate in each of several time periods, not necessarily of equal length, the two equations

$$C_i/N_{i+1} = F_i[\exp(F_i+M_i) - 1]/(F_i+M_i) \qquad (1)$$

$$N_i = N_{i+1} \exp(F_i+M_i) \qquad (2)$$

where

C_i = the catch in time period i,
M_i = the natural mortality rate in time period i,
F_i = the fishing mortality rate in time period i, and
N_i = the stock size at the beginning of time period i.

In each of the time periods the catch and natural mortality must be known. In addition the fishing mortality rate F_t, in the last time period t for which data are available, must be estimated. Then one can solve backward in time to obtain the fishing mortality rates and stock sizes in previous time periods.

To clarify the terminology used herein, the phrase "unevenly distributed catches" will be taken to mean that the intra-year frequency distribution of catches is not constant. Additional symbols and terminology will be introduced below as the need arises.

Pope (1972) has investigated the errors arising from choosing an incorrect value for F_t. Ulltang (1977) has studied various other sources of errors including migration and incorrect values of the natural mortality rate. In addition, he considered the relative error in stock size N_i in year i induced if VPA is carried out using yearly data with the attendant assumption that the catch is evenly distributed throughout the year when, in fact, the fishery is seasonal. However, he assumed that the stock size N_{i+1} at the start of year $i+1$ is known exactly and only considered the error in a single year. Furthermore, he assumed that fishing took place in either the first or last quarter of the year. A further investigation, the results of which are reported herein, of the error in VPA stemming from unevenly distributed catches was carried out using both analytical methods and computer simulation. The analytic techniques are developed first, and then some computer simulations are presented. Finally, the implications for a heavily exploited seasonal fishery with high natural mortality are discussed.

Effects of unevenly distributed catches

Some preliminary assumptions that will simplify the subsequent development and some additional notation are now introduced. It will be assumed that the natural

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mortality rate, M, is a constant yearly rate. Also, it will be assumed that the year can be divided into time intervals of equal length for which catch data are available.

The new notation needed is as follows:

$C_{j,i}$ = the catch in time interval j of year i,
$F_{j,i}$ = the fishing mortality rate in time interval j of year i,
$N_{j,i}$ = the stock size at the start of time interval j of year i,
Δt = the length of each time interval,
k = the number of time intervals in each year,
N_i = the stock size at the start of year i obtained by a VPA carried out on time intervals of length Δt,
$\hat{N}_i$ = the stock size at the start of year i obtained by a VPA carried out on a yearly basis,
$r(N_i) = (\hat{N}_i - N_i)/N_i$ = the relative error in stock size.

Finally, for the sake of convenience, the equal sign will be used when approximations are intended.

Using Pope's estimate (Pope, 1972) the stock size at the beginning of time interval j in year i can be written as

$$N_{j,i} = N_{j+1,i} \exp(M\Delta t) + C_{j,i} \exp(M\Delta t/2).$$

This equation can be applied k times to obtain the following relationship between N_i and N_{i+1},

$$N_i = N_{i+1} \exp(M) + \sum_{j=1}^{k} C_{j,i} \exp[(2j-1)M\Delta t/2] \quad (3)$$

If

$$C_i = \sum_{j=1}^{k} C_{j,i}$$

is the catch in year i which is assumed to be evenly distributed throughout the year, then Pope's estimate gives

$$\hat{N}_i = \hat{N}_{i+1} \exp(M) + C_i \exp(M/2). \quad (4)$$

Using Equations (3) and (4) an expression for the relative error, $r(N_i)$, in terms of $r(N_{i+1})$ can be written down.

Indeed, with

$$F_i = \sum_{j=1}^{k} F_{j,i},$$

$$r(N_i) = [(\hat{N}_{i+1} - N_{i+1})/N_i]\exp(M) + (C_i/N_i) \times$$

$$\times \exp(M/2) - \sum_{j=1}^{k} (C_{j,i}/N_i)\exp[(2j-1)M\Delta t/2]$$

$$= r(N_{i+1})\exp(-F_i) + \sum_{j=1}^{k} (C_{j,i}/N_i)\{\exp(M/2) -$$

$$- \exp[(2j-1)M\Delta t/2]\}$$

For small values of M the second term above can be approximated, using the Maclaurin expansion for the exponential function to give

$$r(N_i) = r(N_{i+1})\exp(-F_i) + \sum_{j=1}^{k} (C_{j,i}/N_i) \times \quad (5)$$

$$\times \{M/2 - [(2j-1)/2](M/k)\}$$

$$= r(N_{i+1})\exp(-F_i) + (M/N_i) \sum_{j=1}^{p} [(k-2j+1)/2k] \times$$

$$\times (C_{j,i} - C_{k-j+1,i})$$

where $p = k/2$ if k is even and $p = (k-1)/2$ if k is odd.

A measure of the intra-year distribution of catches is defined at follows:

$$B_i = k/2 - [\sum_{j=1}^{k} (j-0\cdot5)C_{j,i}]/C_i \quad (6)$$

$$= \{\sum_{j=1}^{p} [(k-2j+1)/2](C_{j,i} - C_{k-j+1,i})\}/C_i$$

where $p = k/2$ if k is even and $p = [(k-1)/2]$ if k is odd.

This quantity can be thought of as a measure of how far the balance point of the sequence of intra-year catches, assumed to be concentrated in the middle of each time period, is from the centre of the interval $(0, k)$ (middle of the year). For want of a better term it will be referred to as the balance.

It is easy to demonstrate that the balance is bounded. Indeed it is shown in the Appendix that

$$-(k-1)/2 \leq B_i \leq (k-1)/2 \quad (7)$$

The similarity of Equation (6) to the second term on the right hand side of Equation (5) allows that equation to be rewritten in the form

$$r(N_i) = r(N_{i+1})\exp(-F_i) + B_i\, E_i\, M/k \quad (8)$$

where $E_i = C_i/N_i$ is the rate of exploitation.

In the simulations discussed below it is assumed that the relative error in stock size in the last year, $r(N_t)$, is equal to zero so that

$$r(N_{t-1}) = B_{t-1}\, E_{t-1}\, M/k.$$

Thus, it might be supposed that, with values for the three factors on the right hand side in a reasonable range, the relative error in stock size is never intolerably large. However, as the VPA proceeds backward in time the error can accumulate. Indeed, if $r(N_t) = 0$, the relative error in year i can be expressed as

$$r(N_i) = (M/kN_i) \sum_{j=i}^{t-1} C_j B_j \exp[(j-i)M]. \qquad (9)$$

This formula is derived in the Appendix.

Therefore, in the absence of any knowledge as to the magnitude of B_i, Equation (7) can be combined with Equation (9) to provide an approximate bound for the magnitude of the relative error in the form;

$$|r(N_i)| \leq [(k-1)M/2kN_i] \sum_{j=i}^{t-1} C_j \exp[(j-i)M]. \qquad (10)$$

It is clear from Equation (9) that if the balance of the catches in each year is negligible, and if $r(N_t) = 0$, then there will be little or no error. A fishery in which the catch is distributed symmetrically throughout the year provides a ready example in which the balance is indeed zero.

Examples and discussion

Many simulations were performed to supplement the analytic results obtained above. Those presented here were carried out assuming that catch data were available on a monthly basis, that is, the year was assumed to be divided into twelve time periods of equal length. In each example a random number generator was used to generate a sequence of monthly fishing mortality rates for a single cohort over a 10-year period. Using these values, sequences of monthly catches and stock sizes at the start of each month were calculated. The resulting stock sizes at the beginning of each year were taken to be the "true" stock sizes. Then a VPA was made on a yearly basis using the true stock size, N_t, at the start of the last year, thereby removing any error due to an incorrect choice of fishing mortality rate in the last year. The results of these simulations are recorded in Tables 1 to 5.

In Tables 1, 2, and 3, the catch was assumed to have taken place in the first quarter of the year, resulting in a balance of 4·5. In the simulation presented in Table 1, a high natural mortality rate of 0·6 was used, and relatively high rates of exploitation were chosen by constraining the monthly fishing mortality rates to lie between 0·03 and 0·17 in the first quater. As expected a combination of high values of the balance, natural mortality, and rates of exploitation led to a large relative error in stock size of approximately 20 %. Note that the error accumulated as the VPA proceeded backward in time. In Table 2 a smaller value of natural mortality of 0·3 was chosen, with the same bounds imposed on the monthly fishing mortality rates, to indicate that the relative error in stock size can remain fairly large in the presence of a high balance. In this case the relative error in stock size builds up to approximately 10 %. To illustrate the effect of smaller rates of exploitation on the relative error in stock size the monthly fishing mortality rates were limited to a range of 0·0 to 0·07. Here with the smaller rates of exploitation, but with a high natural mortality of 0·6, the relative error increases to about 13 % as indicated in Table 3.

To illustrate that unevenly distributed catches need not lead to unacceptably large relative errors in stock size, even in the presence of a high natural mortality rate of 0·6, the catch was assumed to have taken place in the second quarter of the year giving a balance of 1·5. The same restrictions on the monthly fishing mortality rates as used in Table 1 were applied. As is indicated in Table 4 the relative error accumulated to only about 4 %.

In Table 5 the worst possible distribution of monthly catches was assumed, that is, all of the catch was assumed to have occurred in the last time period of the year which gave the maximum, in magnitude, balance of –5·5. The last month of the year was selected, rather than the first, along with a high natural mortality rate of 0·6 to illustrate the fact that Equation (9) loses accuracy when both M and j are relatively large. The use of the Maclaurin expansion through linear terms to approximate the exponential function is the cause for the

Table 1. Results of simulation with balance of 4·5 and natural mortality equal to 0·6 under relatively heavy exploitation.

Age	Catch	Monthly VPA: Fishing mortality	Monthly VPA: Stock size	Yearly VPA: Fishing mortality	Yearly VPA: Stock size	% error in stock size: Actual error	% error in stock size: Estimated error
1	104 604·0	0·1197	1 000 000·0	0·1227	1 197 489·0	19·7	19·2
2	89 403·0	0·2206	486 898·0	0·2260	581 317·0	19·4	18·9
3	43 501·0	0·2470	214 306·0	0·2542	254 511·0	18·8	18·5
4	11 738·0	0·1483	91 871·0	0·1543	108 327·0	17·9	17·8
5	14 262·0	0·4364	43 472·0	0·4526	50 950·0	17·2	17·3
6	4 981·0	0·4280	15 420·0	0·4530	17 782·0	15·3	15·3
7	1 575·0	0·3679	5 516·0	0·4019	6 204·0	12·5	12·4
8	618·0	0·3827	2 095·0	0·4356	2 278·0	8·7	8·6
9	102·0	0·1511	784·0	0·1818	809·0	3·1	2·9
10	135·0	0·4996	370·0	0·6354	370·0	0·0	0·0

Table 2. Results of simulations with balance of 4·5 and natural mortality equal to 0·3 under relatively heavy exploitation.

Age	Catch	Monthly VPA Fishing mortality	Monthly VPA Stock size	Yearly VPA Fishing mortality	Yearly VPA Stock size	% error in stock size Actual error	% error in stock size Estimated error
1	108 444·0	0·1195	1 000 000·0	0·1210	1 097 809·0	9·8	9·9
2	125 034·0	0·2200	657 403·0	0·2228	720 567·0	9·6	9·8
3	82 109·0	0·2460	390 846·0	0·2499	427 187·0	9·3	9·6
4	29 926·0	0·1476	226 393·0	0·1508	246 497·0	8·9	9·2
5	48 997·0	0·4331	144 697·0	0·4419	157 045·0	8·5	8·9
6	23 100·0	0·4231	69 514·0	0·4365	74 785·0	7·6	7·9
7	9 862·0	0·3617	33 732·0	0·3795	35 807·0	6·2	6·4
8	5 225·0	0·3735	17 405·0	0·4004	18 150·0	4·3	4·4
9	1 169·0	0·1471	8 876·0	0·1620	9 009·0	1·5	1·5
10	2 072·0	0·4763	5 676·0	0·5395	5 676·0	0·0	0·0

Table 3. Results of simulation with balance of 4·5 and natural mortality equal to 0·6 under relatively light exploitation.

Age	Catch	Monthly VPA Fishing mortality	Monthly VPA Stock size	Yearly VPA Fishing mortality	Yearly VPA Stock size	% error in stock size Actual error	% error in stock size Estimated error
1	49 579·0	0·0550	1 000 000·0	0·0600	1 128 585·0	12·9	11·9
2	37 079·0	0·0801	519 464·0	0·0879	583 305·0	12·3	11·4
3	20 243·0	0·0866	263 140·0	0·0958	293 181·0	11·4	10·6
4	7 381·0	0·0620	132 432·0	0·0693	146 198·0	10·4	9·6
5	7 902·0	0·1332	68 311·0	0·1500	74 867·0	9·6	8·9
6	3 723·0	0·1305	32 813·0	0·1496	35 363·0	7·8	7·2
7	1 591·0	0·1149	15 805·0	0·1344	16 711·0	5·7	5·3
8	794·0	0·1174	7 732·0	0·1402	8 018·0	3·7	3·4
9	208·0	0·0613	3 773·0	0·0748	3 825·0	1·4	1·2
10	238·0	0·1413	1 948·0	0·1756	1 948·0	0·0	0·0

Table 4. Results of simulation with balance of 1·5 and natural mortality of 0·6.

Age	Catch	Monthly VPA Fishing mortality	Monthly VPA Stock size	Yearly VPA Fishing mortality	Yearly VPA Stock size	% error in stock size Actual error	% error in stock size Estimated error
1	90 033·0	0·1197	1 000 000·0	0·1212	1 042 609·0	4·3	5·5
2	76 950·0	0·2206	486 898·0	0·2227	506 882·0	4·1	5·4
3	37 441·0	0·2470	214 306·0	0·2496	222 639·0	3·9	5·3
4	10 103·0	0·1483	91 871·0	0·1509	95 197·0	3·6	5·1
5	12 276·0	0·4365	43 473·0	0·4394	44 927·0	3·3	5·0
6	4 287·0	0·4280	15 420·0	0·4327	15 889·0	3·0	4·4
7	1 335·0	0·3677	5 516·0	0·3750	5 657·0	2·6	3·6
8	532·0	0·3827	2 096·0	0·3933	2 134·0	1·8	2·5
9	88·0	0·1515	784·0	0·1589	790·0	0·7	0·8
10	116·0	0·4986	370·0	0·5215	370·0	0·0	0·0

Table 5. Results of simulations with balance of $-5 \cdot 5$ and natural mortality equal to $0 \cdot 6$.

Age	Catch	Monthly VPA Fishing mortality	Monthly VPA Stock size	Yearly VPA Fishing mortality	Yearly VPA Stock size	% error in stock size Actual error	% error in stock size Estimated error
1	63 330·0	0·1193	1 000 000·0	0·1164	762 167·0	−23·8	−19·1
2	54 082·0	0·2195	487 082·0	0·2121	372 332·0	−23·6	−18·9
3	26 309·0	0·2454	214 623·0	0·2347	165 283·0	−23·0	−18·4
4	7 105·0	0·1472	92 157·0	0·1402	71 730·0	−22·2	−17·7
5	8 613·0	0·4308	43 652·0	0·3979	34 217·0	−21·6	−17·2
6	3 008·0	0·4195	15 572·0	0·3730	12 615·0	−19·0	−15·2
7	952·0	0·3576	5 618·0	0·3032	4 768·0	−15·1	−12·1
8	373·0	0·3666	2 156·0	0·2918	1 932·0	−10·4	−8·4
9	62·0	0·1442	820·0	0·1093	792·0	−3·4	−2·8
10	81·0	0·4599	390·0	0·3176	390·0	0·0	0·0

loss of accuracy. To mitigate somewhat this circumstance, the equation

$$r(N_i) = (M/kN_i)\exp(M/2) \sum_{j=1}^{i-1} C_j B_j \exp[(j-i)M] \qquad (11)$$

was used in place of Equation (8) to estimate the relative error in stock size. Equation (11) is obtained by factoring out $\exp(M/2)$ before deriving Equation (5) and thence Equation (9). As can be seen in Table 5, however, the approximate relative error given by Equation (11) underestimates, in magnitude, the true relative error. Again it is apparent that high natural mortality along with a large value of the balance can lead to an unacceptably large relative error in stock size.

From Tables 1 to 5 it is apparent that if the majority of the catch is taken at the beginning of the year then a yearly VPA yields an overestimate of the stock size, whereas an underestimate is provided by the yearly VPA if the preponderance of the catch occurs in the latter part of the year.

Although the thrust of this study was to determine relative errors in stock size, it is nevertheless worth noting that the difference between the yearly fishing mortality rate and the sum of the monthly rates tends to zero in each of the simulations.

It is clear from the results in this section as well as those in the previous section that catches unevenly distributed throughout the year can lead to significantly large relative errors in stock sizes. However, the uneven distribution of catches is not a significant problem unless the balance is high (that is, the majority of the catch occurs at one end of the year or the other), and the natural mortality and/or the rate of exploitation are high. For instance, if a fishery is seasonal with heavy exploitation at one end of the year or the other and if natural mortality is high then the relative error in stock size can accumulate to a quite large value.

Conclusions

Both the analysis and simulations indicate that the effects of unevenly distributed catches on the relative error in stock size when a VPA is carried out with the supposition of catches evenly distributed throughout the year are not severe unless the natural mortality is large and/or the fishery is heavily exploited. However, if the fishery is seasonal with heavy exploitation at one end of the year or the other and if the natural mortality is high, then the relative error in stock size can be unacceptably large. If these conditions prevail in a fishery it is recommended that the VPA be performed, if possible, on the basis of time units smaller than a year.

Acknowledgements

M. L. Parrack of the Galveston Laboratory, Southeast Fisheries Center, National Marine Fisheries Service, Miami, Florida, suggested this problem and provided assistance in the preparation of this manuscript.

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Appendix

Proof of Inequality (7)

Since $1 \leq j \leq k$ and $C_{j,i} \geq 0$ it is clear that

$$(2k-1)C_{j,i} \geq (2j-1)C_{j,i} \geq C_{j,i}$$

Thus

$$(2k-1)\sum_{j=1}^{k} C_{j,i} \geq \sum_{j=1}^{k} (2j-1)C_{j,i} \geq \sum_{j=1}^{k} C_{j,i}.$$

Recalling that

$$C_i = \sum_{j=1}^{k} C_{j,i}$$

and dividing each member by C_i gives

$$(2k-1) \geq [\sum_{j=1}^{k} (2j-1)C_{j,i}]/C_i \geq 1.$$

Multiplying through by $-1/2$ and adding $k/2$ to each member yields Inequality (7).

Derivation of Equation (9)

Equation (8), with $r(N_t) = 0$ and $i = t-1$ gives

$$r(N_{t-1}) = B_{t-1}\, E_{t-1}\, M/k.$$

Hence, another application of Equation (8) yields

$$\begin{aligned} r(N_{t-2}) &= r(N_{t-1})\exp(-F_{t-2}) + B_{t-2}\, E_{t-2}\, M/k \\ &= (B_{t-1}\, E_{t-1}\, M/k)\exp(-F_{t-2}) + \\ &\quad + B_{t-2}\, E_{t-2}\, M/k \\ &= (M/k)\{[C_{t-1}\exp(M)/N_{t-1} \times \\ &\quad \times \exp(F_{t-2}+M)] + B_{t-2}\, E_{t-2}\} \\ &= (M/kN_{t-2})[B_{t-1}\, C_{t-1}\exp(M) + \\ &\quad + B_{t-2}\, C_{t-2}]. \end{aligned}$$

Repetition of this argument produces Expression (9).